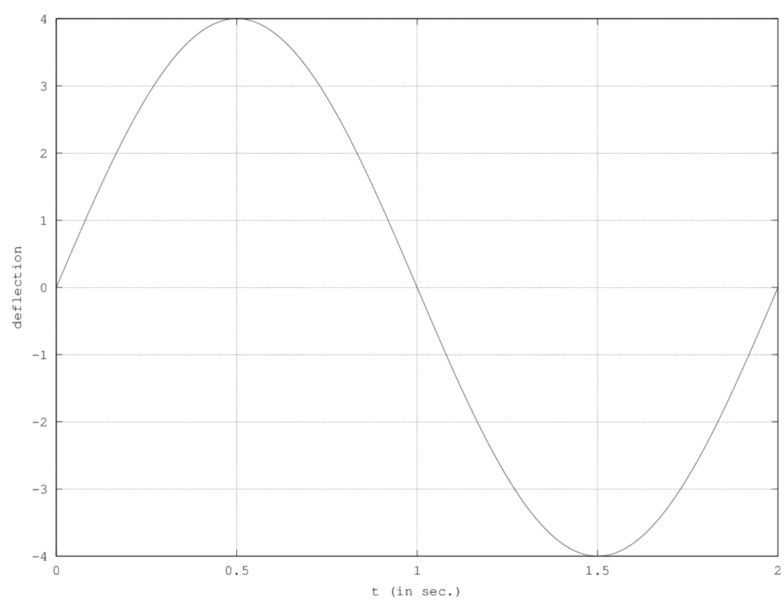
Exam Signals and Systems

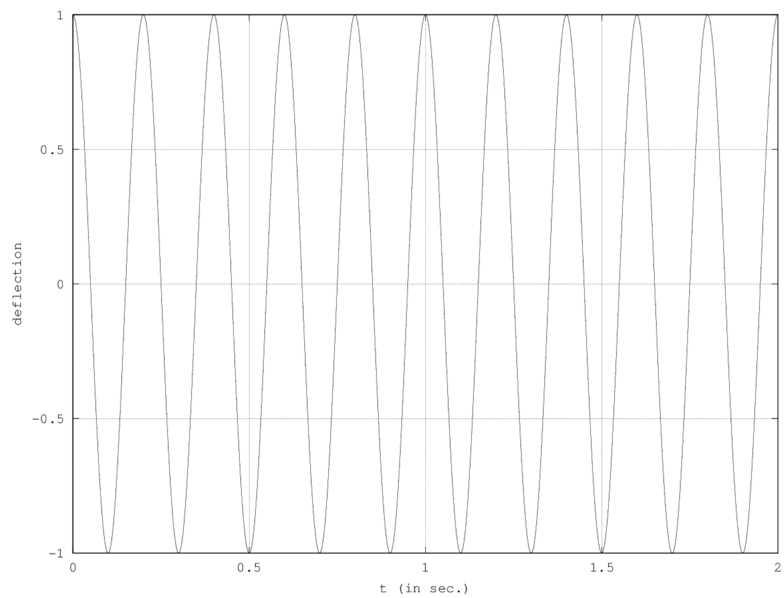
22 januari 2015, 9:00-12:00

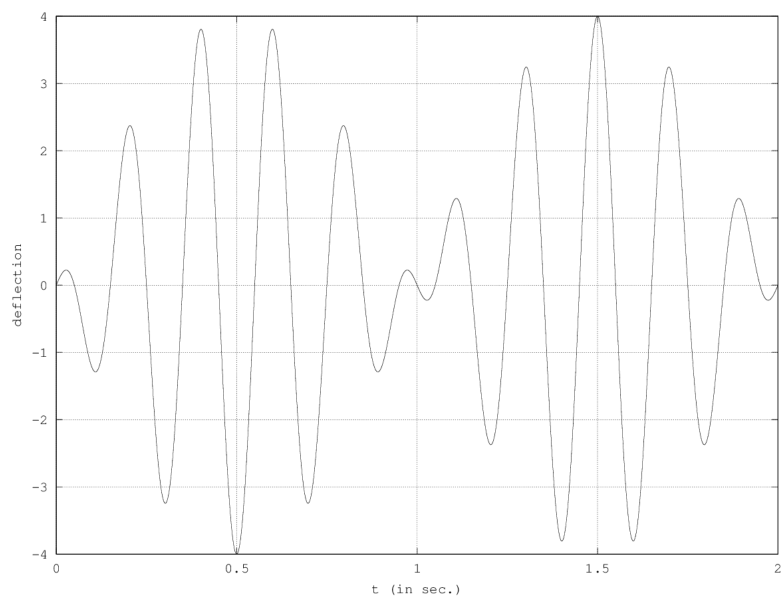
- Please put your student identification card on your desk.
- Write your name, student number and the total number of sheets on the first sheet. Number the sheets!
- You may answer the problems in Dutch or in English.
- Please read each problem fully before making it. Write neatly and carefully. If the handwriting is unreadable, or needs guessing to make something out of it, then the answer is rejected.
- This exam comes with a formula sheet. If you are using a formula from this sheet, please indicate the number of the formula. Other literature, such as the book, may not be consulted.
- The use of a (graphical) calculator is permitted.
- For answers without explanation (even if the answer is correct) no points are awarded.
- The exam consists of five problems. Problem 1 is worth 16 points, problem 2 is worth 14 points, and the remaining problems are worth 20 points each. You get 10 points for free (total 100 points).

Problem 1: signals and spectra

The figures below show three continuous signals (deflection as a function of the time).







(a) Determine for each signal a formula for the deflection as a function of the time.

(b) Given the continuous signals $x(t)$, $y(t)$ and $z(t)$

$$\begin{aligned} x(t) &= -\cos(8\pi t) \\ y(t) &= 4\sin(6\pi t) \\ z(t) &= x(t)y(t) \end{aligned}$$

Write each of these signals as a **sum** of terms of the form $Xe^{j\alpha t}$.

(c) Plot the spectrum of each signal. Show the units along the axes and specify for each frequency component the corresponding phase angle.

(d) Derive a formula for the chirp signal $x(t)$ of which the frequency sweeps linearly from $f_0 = 220$Hz to $f_1 = 2320$Hz over a 3-second time interval. At $t = 0$ the phase angle is $\phi = 0$, and the deflection is 2.

Problem 2: Instantaneous frequency, spectrograms, and sampling
Given are the continous-time signals

$$\begin{aligned} x(t) &= \cos(200\pi t + 100\pi t^2) \\ y(t) &= 6\cos(30\pi t + \pi/2) + \cos(15\pi t + \pi/2) \end{aligned}$$

(a) Give an expression for the *instantaneous frequency* (in Hz) of $x(t)$ as a function of time.

(b) Sketch the spectrogram of $x(t)$ for $0 \le t \le 2$. Make sure to label the axes. [Note: the frequency axis must be in Hz]

(c) The signal $y(t)$ is sampled with a sampling rate of 60 samples per second using a perfect A/D converter to create the discrete time signal $y[n]$. Give an expression for $y[n]$ in terms of the resulting dicrete-time frequencies ($\hat{\omega}$).

(d) Does aliasing occur in the setting as described in (c)? Explain your answer.

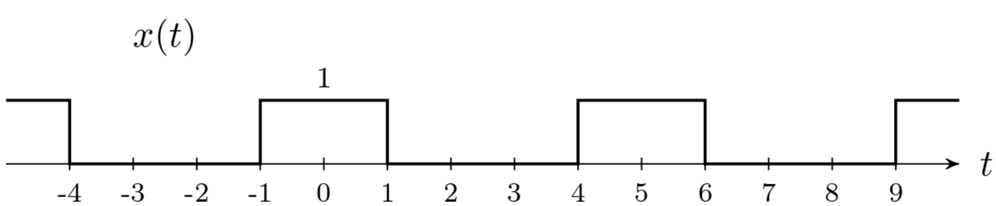
Problem 3: Fourier analysis

(a) The Fourier coefficients of a continuous signal (with period $T = \frac{1}{100}$ sec.) are:

$$a_k = \begin{cases} 2 & \text{for } k = -5 \\ j & \text{for } k = -1 \\ 3 & \text{for } k = 0 \\ -j & \text{for } k = 1 \\ 2 & \text{for } k = 5 \\ 0 & \text{for all other } k \end{cases}$$

This signal can be written in the form $DC + A\cos(2\pi f_0 t + \phi_0) + B\cos(2\pi f_1 t + \phi_1)$. Determine the values of DC, A, f_0, ϕ_0, B, f_1 and ϕ_1.

(b) The following figure shows the periodic continuous signal $x(t)$ with period $T = 5$ seconds.



Show that the Fourier coefficients a_k of the signal $x(t)$ are given by

$$a_k = \begin{cases} 2/5 & \text{for } k = 0 \\ \frac{j}{2\pi k}(e^{-j2\pi k/5} - e^{-j8\pi k/5}) & \text{for } k \neq 0 \end{cases}$$

(c) Determine the Fourier coefficients b_k of the signal $y(t) = x(t-1)$.

(d) Determine the Fourier coefficients of the function $z(t) = 5 + 2\sin(2\pi 120 t)\cos(2\pi 30 t)$.

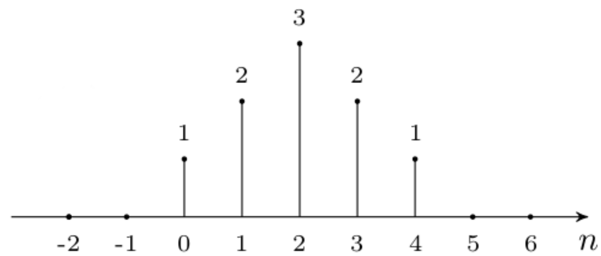
Problem 4: LTI-systems

(a) Two systems F_0 and F_1 are governed by the equations

$$\begin{aligned} y_0[n] &= (x[n-1])^2 \\ y_1[n] &= 2x[-n] \end{aligned}$$

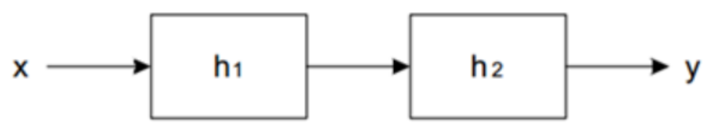
where $x[n]$ is the input and $y_{0,1}[n]$ are the outputs. For both systems, determine whether they are *causal*, *linear*, and *time invariant*.

(b) The impulse response $h[n]$ of a discrete-time LTI system is shown in the following figure.



What is the output of this system when we feed it on its input the unit-step function $x[n] = u[n]$? [Note: $u[n] = 0$ for $n < 0$, and $u[n] = 1$ for $n \geq 0$]

(c) Consider the following system that is composed of two cascaded LTI-systems.

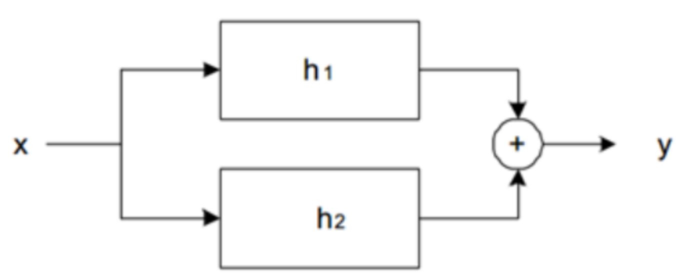


The impulse response $h_1[n]$ is given as $h_1[n] = \delta[n] + \delta[n-1] + \delta[n-2]$. We feed the system with the discrete time signal $x[n] = \delta[n] + \delta[n-2]$, and observe the output

$$y[n] = \delta[n] + 2\delta[n-1] + 4\delta[n-2] + 4\delta[n-3] + 4\delta[n-4] + 2\delta[n-5] + \delta[n-6]$$

Determine the impulse response $h_2[n]$.

(d) Consider the following system that is composed of two parallel LTI-systems.



The impulse response $h_1[n]$ is given as $h_1[n] = \delta[n] + \delta[n-1] + \delta[n-2]$.
We feed the system with the discrete time signal $x[n] = \delta[n] + \delta[n-2]$, and observe the output

$$y[n] = \delta[n] + 3\delta[n-1] + 3\delta[n-2] + 3\delta[n-3] + 2\delta[n-4]$$

Determine the impulse response $h_2[n]$.

Problem 5: frequency responses and z-transforms

(a) Give the z-transform (or system function) $H(z)$ and the frequency response $H(e^{j\hat{\omega}})$ of a 5-point running average filter.

(b) The z-transform (or system function) of a FIR system is given by $H(z) = 1 - 2\cos(\hat{\omega})z^{-1} + z^{-2}$. Determine the roots of $H(z)$, and use the result to determine the output of the system when we feed it the input $x[n] = 1 + 3\sin(n\hat{\omega})$.

(c) Construct the system function $H(z)$ and the difference equation of a system that removes the signal $x[n] = 1 + \cos(\frac{\pi n}{3})\cos(\frac{\pi n}{4})$ completely. Of course, the trivial solution $H(z) = 0$ is not allowed.

(d) Let H_1 be some FIR system. Use the z-transform to show that there cannot exist a FIR system H_2 which is the inverse of H_1 (in other words, if we connect the output of H_1 to the input of H_2, then the resulting cascaded system would be the identity function).

Formula sheet Signals and Systems

$$\cos(\theta) = \sin(\theta + \pi/2) \tag{1}$$
$$\cos(\theta) = \cos(\theta + 2\pi k) \text{ for integer } k \tag{2}$$
$$\cos(\theta) = \cos(-\theta) \tag{3}$$
$$\sin(-\theta) = -\sin(\theta) \tag{4}$$
$$\cos(2\pi k) = 1 \text{ for integer } k \tag{5}$$
$$\cos(\pi k + \frac{\pi}{2}) = 0 \text{ for integer } k \tag{6}$$
$$\cos(2\pi k + \pi) = -1 \text{ for integer } k \tag{7}$$
$$\cos^2\theta + \sin^2\theta = 1 \tag{8}$$
$$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2} \tag{9}$$
$$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2} \tag{10}$$
$$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2} \tag{11}$$
$$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2} \tag{12}$$
$$j^2 = -1 \tag{13}$$
$$Re(a+jb) = a \tag{14}$$
$$Im(a+jb) = b \tag{15}$$
$$(a+jb)^* = a - jb \tag{16}$$
$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{17}$$
$$\cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2} \tag{18}$$
$$\sin(\theta) = \frac{e^{j\theta}-e^{-j\theta}}{2j} \tag{19}$$
$$\int e^{\alpha t}dt = \frac{1}{\alpha}e^{\alpha t} + c \tag{20}$$
$$\int te^{\alpha t}dt = \frac{\alpha t - 1}{\alpha^2}e^{\alpha t} + c \tag{21}$$
$$x[n] = x(nT_s) \text{ Perfect A-to-D conversion} \tag{22}$$
$$f_s = \frac{1}{T_s} \text{ Sampling frequency} \tag{23}$$
$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s} \text{ Normalized Radian Frequency} \tag{24}$$
$$y(t) = \sum_{n=-\infty}^{\infty} y[n]p(t - nT_s) \text{ Interpolation/Reconstruction} \tag{25}$$
$$p(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}, \quad -\infty < t < \infty \text{ Sinc pulse} \tag{26}$$
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \text{ Fourier synthesis} \tag{27}$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt \text{ Fourier analysis} \tag{28}$$

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] \text{ FIR system} \tag{29}$$

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \text{ Unit impulse response} \tag{30}$$

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] \text{ Convolution sum FIR-system} \tag{31}$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \text{ General convolution} \tag{32}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k} \text{ Frequency response FIR-system} \tag{33}$$

$$h_1[n] * h_2[n] \leftrightarrow H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}}) \tag{34}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)} \text{ Dirichlet function} \tag{35}$$

$$X(z) = \sum_{k=0}^{N} x[k]z^{-k} \text{ Z-transform} \tag{36}$$

$$H(z) = \sum_{k=0}^{N} b_k z^{-k} = \sum_{k=0}^{N} h[k]z^{-k} \text{ System function FIR system} \tag{37}$$

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z) \text{ Linearity of z-transform} \tag{38}$$

$$y[n] = h[n] * x[n] \leftrightarrow Y(z) = H(z)X(z) \text{ Convolution via z-domain} \tag{39}$$

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{40}$$